BASIC NOTIONS AND RESULTS IN GRAPH THEORY. (MATH 240 REVIEW)

Definition of a graph. A *graph* G is an ordered pair $(V(G), E(G))$, where $V(G)$ is a set of *vertices*, $E(G)$ is a set of some pairs of vertices called *edges*.[1] We will write uv, instead of $\{u, v\}$, to denote the edge consisting of a pair of vertices u and v for brevity. For example, if

$$V(G) = \{a, b, c, d\},$$
$$E(G) = \{ab, ac, bc, cd\},$$

then G is a graph with four vertices and four edges.

The vertices u and v are called *the ends* of the edge $e = uv$, and the edge e is *incident* to its ends. Two vertices u and v are *adjacent* or *neighbors* if $uv \in E(G)$.

Standard graph classes. A *complete graph* on n vertices is denoted K_n is a simple graph in which every two vertices are adjacent.

A *path* on n vertices, denoted P_n, is a graph such that:

$$V(P_n) = \{v_1, v_2, \dots, v_n\}$$
$$E(P_n) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n\}.$$

The vertices v_1 and v_n are *the ends* of the path P_n.

A *cycle* on $n \geq 3$ vertices, denoted C_n, is a graph such that:

$$V(C_n) = \{v_1, v_2, \dots, v_n\}$$
$$E(C_n) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1\}.$$

Degrees. The *degree* of a vertex v in a graph G is the number of its neighbors. It is denoted by $\deg_G(v)$, or usually simply $\deg(v)$ when the graph G is understood from context.

Theorem 1. *For any graph G, we have:*

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)|$$

Subgraphs. A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We write $H \subseteq G$ to denote that H is a subgraph of G.

A *path*(or *cycle*) of G is a subgraph of G which is a path (or resp. cycle).

A subgraph $G \setminus e$ obtained from G *by deleting the edge* $e \in E(G)$ is defined by $V(G \setminus e) = V(G)$, $E(G \setminus e) = E(G) - \{e\}$. A subgraph $G \setminus v$ of G obtained from G *by deleting the vertex* v is defined by $V(G \setminus v) = V(G) - \{v\}$, and $E(G \setminus v)$ consists of all the edges of G not incident to v.

Isomorphism. Graphs H and G are *isomorphic* if there exists a bijection $\phi : V(H) \to V(G)$ (called *an isomorphism*) such that $uv \in E(H)$ if and only if $\phi(u)\phi(v) \in E(G)$. We frequently treat isomorphic graphs as being the same.

[1] Unlike other definitions this one does not allow for loops and parallel edges.

Connectivity. A *walk from v_0 to v_k* in a graph G is a non-empty alternating sequence $v_0, e_0, v_1, e_1, \ldots, e_{k-1}, v_k$ of vertices and edges in G, such that $e_i = v_i v_{i+1}$ for $i = 0, \ldots, k-1$. If $v_0 = v_k$, then the walk is said to be *closed*.

A graph G is *connected* if and only if for all $u, v \in V(G)$, there exists a walk from u to v.

Lemma 2. *If there is a walk with ends u, v in G, then there is a path in G with the same ends.*

A *connected component* of a graph G is a maximal connected subgraph.

Lemma 3. *In a graph G, every vertex is in a unique connected component.*

Let $\mathrm{comp}(G)$ denote the number of components of a graph G.

Trees and Forests. A *forest* is a graph with no cycles. A *tree* is a connected forest.[2]

Theorem 4. *If G is a (non-null) forest, then*

$$\mathrm{comp}(G) = |V(G)| - |E(G)|.$$

In particular, if G is a tree then

$$|V(T)| = |E(T)| + 1$$

A vertex v in a tree with $\deg(v) = 1$ is called a *leaf*.

Lemma 5. *Let T be a tree with $|V(T)| \geq 2$. Then, T has at least twp leaves, and if T has exactly two leaves, then T is a path.*

Spanning trees. Let G be a graph, $T \subseteq G$ a tree, with $V(G) = V(T)$. Then, T is called a *spanning tree* of G.

Lemma 6. *Let G be a connected graph. Let H be a subgraph of G such that either*

- *H is minimal such that $V(H) = V(G)$ and H is connected, or*
- *H is maximal such that H contains no cycles.*

Then H is a spanning tree of G.

Euler's Theorem and Hamiltonian Cycles.

Theorem 7 (Euler's Theorem)**.** *Let G be a connected graph in which every vertex has even degree. Then there exists a closed walk in G using every edge exactly once.*

A cycle C in a graph G is *Hamiltonian* if $V(G) = V(C)$.

Theorem 8 (Dirac)**.** *Let G be a simple graph with $|V(G)| \geq 3$. If $\deg(v) \geq |V(G)|/2$ for every $v \in V(G)$ then G has a Hamiltonian cycle.*

Bipartite Graphs. A *bipartition* of a graph G is a pair of subsets (A, B) of $V(G)$ so that $A \cap B = \emptyset$, $A \cup B = V(G)$, and every edge of G has one end in A and another in B. A graph is *bipartite* if it admits a bipartition.

Theorem 9. *For every graph G, the following statements are equivalent:*

(1) *G is bipartite,*
(2) *G has no closed walk with odd number of edges,*
(3) *G has no odd cycle.*

[2] If one is being pedantic, then one should note that the *null graph* with no vertices and no edges is considered to be a forest, but not a tree.

Vertex coloring. Let G be a graph, S a set of size k. The function $\varphi : V(G) \to S$ is called a *(proper) k-coloring* if for all $e \in E(G)$ with ends u and v, $\varphi(u) \neq \varphi(v)$. Elements of S are called *colors*. The set of all vertices of the same color is called a *color class*.

The *chromatic number* $\chi(G)$ is the minimum k such that G admits a k-coloring.

Let $\Delta(G)$ denote the maximum degree of a vertex in a graph G.

Theorem 10. $\chi(G) \leq \Delta(G) + 1$ *for every graph G.*

Planar graphs. A *drawing* of a graph G in the plane represents vertices as points in the plane, and edges as curves which do not intersect themselves or each other, and have ends at points corresponding to the ends of edges. Points in the plane not used in the drawing are divided into *regions*. Two points belong to the same region if they can be joined by a curve which avoids the drawing.

Let $\mathrm{Reg}(G)$ denote the number of regions in the planar drawing of G.

Theorem 11 (Euler's Formula)**.** *Let G be a connected graph drawn in the plane. Then*

$$|V(G)| - |E(G)| + \mathrm{Reg}(G) = 2.$$

Lemma 12. *If G is planar, $|E(G)| \geq 2$, then $|E(G)| \leq 3|V(G)| - 6$. If, further, G contains no K_3 subgraphs, then $|E(G)| \leq 2|V(G)| - 4$.*